Sensor and Simulation Notes

Note 428

21 November, 1998

Impedance of an Azimuthal TEM Waveguide Bend in a Graded Dielectric Medium

W. Scott Bigelow

Everett G. Farr

Farr Research, Inc.

Abstract

A continuously graded dielectric material may be used to compensate a coaxial transmission line bend in order to allow it to support a TEM mode. The characteristic impedance of such a bend is a critical design parameter. Here we derive an expression which can be used to calculate that impedance from a finite element solution of the relevant potential equation. We assume no variation in material properties or geometry in the azimuthal direction, the direction of propagation through the bend. We further assume that the space between perfect inner and outer conductors within the bend is filled with a non-magnetic, linear, isotropic, lossless dielectric material whose permittivity varies inversely as the square of the radius of curvature. We demonstrate our approach by impedance calculations for canonical problems for which there are known solutions. We also consider bends compensated by layered approximations of graded dielectric materials.

1. Introduction

In principle, a continuously graded dielectric material may be used to compensate a coaxial transmission line bend in order to allow it to support a pure TEM mode. The effective impedance of such a bend is a critical design parameter. Here we derive an expression which can be used to calculate that impedance from a finite element solution of the relevant potential equation. In the course of that derivation, we obtain a similar expression applicable to a layered approximation to the continuously graded bend and an expression applicable to a straight transmission line section.

For the continuously graded bend, we assume no variation in material properties or geometry in the azimuthal (j) direction, the direction of propagation through the bend; the *z*-axis is an axis of rotational symmetry. We further assume that the space between inner and outer conductors within the bend is filled with a non-magnetic, linear, isotropic, lossless dielectric material whose relative permittivity varies inversely with the square of the radius of curvature as

$$e_r = \frac{e}{e_0} = \left(\frac{y_{\max}}{y} \right)^2 \tag{1}$$

where y is the radius of curvature in the (y , j , z) coordinate system, and e_0 is the permittivity of free space. This variation compensates for the variation of arc length with the radius of curvature and thus preserves the wavefront as it propagates through the bend. Since the wavefront is preserved, its *angular* propagation velocity, v_j , is independent of the radius of curvature, and we can obtain the characteristic impedance from

$$Z_C = \frac{1}{v_j \, C_j} \tag{2}$$

where C_j , the capacitance per unit azimuth angle, is obtained from the finite element solution.

2. Derivation

We begin our derivation with Maxwell's equation relating the divergence of the electric displacement to the net free charge density,

$$\nabla \cdot \vec{D} = r \tag{3}$$

Since we assume our medium is linear and isotropic, we have $\vec{D} = e\vec{E}$, where $\vec{E}$ is the electric field intensity. For now, we make no assumptions regarding either symmetry or the spatial dependence of the permittivity. We assume that the only charges are those induced by application of an arbitrary voltage, $V = V_0$, to one of the conductors, while the other is grounded ($V = 0$). These are Dirichlet boundary conditions. The conductors, forming the surface, *S*, bound the volume between them, the domain, Ω. Within Ω, the charge density is zero and the electric field is given by the negative of the potential gradient. Thus, the displacement equation, (3), becomes

$$\nabla \cdot \left(e \nabla V \right) = 0 \tag{4}$$

Since we seek an estimate of the capacitance from a finite element solution of (4), we require a functional, *W*(*V*), which is stationary about $V = u$, the correct solution to this equation and its boundary conditions. We multiply (4) by *V* and integrate over Ω to obtain

$$\int_\Omega V \nabla \cdot \left(e \nabla V \right) d\Omega = 0 \tag{5}$$

Now, using the vector identity, $\nabla \cdot (f\vec{A}) = \nabla f \cdot \vec{A} + f \nabla \cdot \vec{A}$, with $f = V$ and $\vec{A} = e \nabla V$, (5) becomes

$$\int_\Omega \left(e \nabla V \right) \cdot \left(\nabla V \right) d\Omega - \int_\Omega \nabla \cdot \left(V e \nabla V \right) d\Omega = 0 \tag{6}$$

Application of the divergence theorem (Green's theorem in space) to the second integral in (6) produces the variational or weak form of the potential equation,

$$\int_\Omega \left(e \nabla V \right) \cdot \left(\nabla V \right) d\Omega = \int_S \hat{n} \cdot \left(e \nabla V \right) V \, dS \tag{7}$$

Recognizing the integral over Ω in (7) as twice the energy stored in the electric field, we write

$$W\left(U\right) = \frac{1}{2} \int_\Omega \left(e \nabla U \right) \cdot \left(\nabla U \right) d\Omega \tag{8}$$

where, if *U* is (merely) continuous in Ω and satisfies (just) the Dirichlet conditions on the boundary, *S*, of Ω, then the functional, *W*(*U*), is stationary about $U = u$, the solution of (4) [1].

We now specialize to an axially symmetric geometry. Assuming no j-dependence, we carry out the integration over j in (8) to obtain the stored energy per unit azimuth angle as

$$W_j(U) = \frac{W(U)}{2p} = \frac{e_0}{2}\int_R (e_r \nabla U)\cdot(\nabla U) y \; dy \; dz \tag{9}$$

where R is a j-plane slice through Ω, and we have factored out the constant permittivity of free space. Since the same stored energy is also expressed by

$$W_j(U) = (1/2) C_j(U) V_0^2 \tag{10}$$

we have for the capacitance per unit angle

$$C_j(U) = \frac{e_0}{V_0^2} I(U) \tag{11}$$

where we have defined the functional

$$I(U) = \int_R (e_r \nabla U)\cdot(\nabla U) y \; dy \; dz \tag{12}$$

If we compare (12) with (8), the functional minimized by the solution of (4), then (12) becomes recognizable as a functional which is minimized by the solution, $U = u(y\;, z)$, of the two-dimensional potential equation for an axially symmetric geometry

$$\boxed{\nabla \cdot (y \; e_r \nabla u) = 0} \tag{13}$$

This is a significant result. Except for the requirement for axial symmetry, the spatial dependence of the permittivity is arbitrary. Thus, solutions of (13) form a more general class than those for which (1) holds. In particular, we later show how (13) can be applied to a structure in which discrete layers of dielectric materials are used to approximate the spatial dependence of (1).

Next, the solution of (13) is expanded in a series of dimensionless basis functions, a_j, as

$$u(y_i, z_i) = \sum_{j=1}^{N} U_j \, a_j(y_i, z_i) \tag{14}$$

The basis functions are defined on a triangular mesh, such that they are continuous in R and satisfy the Dirichlet conditions on the boundary of R. Additionally, they are zero at all nodes except at (y_i, z_i), where they take on the value of unity. This leads to the result that $u(y_i, z_i) = U_i$ the potential at the i-th node. Then, use of (14) in (12) leads to

$$I(u) = \sum_i \sum_j U_i U_j \, S_{ij} \tag{15}$$

where

$$S_{ij} = \int y \;\; e_r \nabla a_i \cdot \nabla a_j \; dy \;\; dz \tag{16}$$

In matrix form, the functional, $I(u)$, may be expressed as

$$I(u) = \mathbf{U}^{\mathbf{T}}\mathbf{S}\mathbf{U} \tag{17}$$

Here, $\mathbf{U}$, the finite element solution of (13), is the vector of coefficients, U_j. The $\mathrm{N} \times \mathrm{N}$ finite element matrix, $\mathbf{S}$, of “energy” inner products is sometimes called the Dirichlet matrix or, in problems of elasticity, the stiffness matrix. In the present case, its dimensions are length. Contained within it are any spatial dependencies of the permittivity.

We can now combine (11) and (17) to obtain the capacitance for an axially symmetric system from the finite element solution of (13) as

$$C_j = \frac{e_0}{V_0^2} \mathbf{U}^{\mathbf{T}}\mathbf{S}\mathbf{U} \tag{18}$$

This is as far as we can proceed rigorously without imposing an additional constraint on the spatial dependence of the permittivity. Apart from the requirement of no j-dependence, (18) admits any well-behaved spatial dependence for the permittivity. Unless (1) is obeyed, however, the propagation velocity will vary across the region, R, and no characteristic impedance in the sense of (2) will be defined.

To proceed further, we must require that the permittivity obey (1), so that the angular propagation velocity will be a constant, and the characteristic impedance will be given by (2). Then, since the propagation velocity, v, at $y = y_{\max}$, is c, the speed of light, we have for the constant angular velocity, $v_j = c/y_{\max}$. Furthermore, since $Z_0 = 1/(e_0 c)$, where Z_0 is the impedance of free space, (2) can be written as

$$Z_C = \frac{Z_0 \, e_0 y_{\max}}{C_j} \tag{19}$$

Using (18), we obtain

$$\boxed{Z_C = Z_0 \left(\frac{y_{\max} V_0^2}{\mathbf{U}^{\mathbf{T}}\mathbf{S}\mathbf{U}} \right)} \tag{20}$$

which is independent of V_0, since $\mathbf{U}$ contains V_0 as a linear factor. Note that the solution of (13) may be used in this expression to obtain an estimate of the characteristic impedance when (1) is true in only an approximate sense, for example, when dielectric layers having different constant permittivities are used to approximate the y dependence in (1).

When (1) is rigorously true, it becomes convenient to obtain alternate forms of some of the expressions introduced above, forms which directly incorporate the analytic form of the permittivity profile. We return to the two-dimensional potential equation for axial symmetry, (13), and to the functional minimized by its solution, (12). Substituting (1) for e_r, we obtain

$$I(U) = y_{\max}^{2} \int_R \left(\frac{1}{y} \nabla U \right) \cdot (\nabla U)\, dy\ dz \qquad (21)$$

for the functional, and

$$\boxed{\nabla \cdot \left(\frac{1}{y} \nabla u \right) = 0} \qquad (22)$$

for the potential equation. This equation is equivalent to the expression derived by Baum in [2, Appendix A]. Its solution minimizes the integral over the region, R, in (21). The elements of the Dirichlet matrix, (16), now become

$$S_{ij} = y_{\max}^{2} K_{ij} \qquad (23)$$

where

$$K_{ij} = \int \frac{1}{y} \nabla a_i \cdot \nabla a_j\, dy\ dz \qquad (24)$$

Thus, (17), the matrix form of $I(u)$, becomes

$$I(u) = y_{\max}^{2} \mathbf{U}^{\mathbf{T}} \mathbf{K} \mathbf{U} \qquad (25)$$

where $\mathbf{K}$ is the assembled matrix of K_{ij} elements, with dimensions of inverse length. The finite element solution of (22), $\mathbf{U}$, is the vector of coefficients, U_j. The capacitance corresponding to the finite element solution, (18), becomes

$$C_j = \frac{e_0 y_{\max}^{2}}{V_0^2} \mathbf{U}^{\mathbf{T}} \mathbf{K} \mathbf{U} \qquad (26)$$

Finally, since the angular velocity of propagation is a constant, we can substitute (26) in (19) to obtain

$$\boxed{Z_C = Z_0 \left(\frac{V_0^2}{y_{\max} \mathbf{U}^{\mathbf{T}} \mathbf{K} \mathbf{U}} \right)} \tag{27}$$

as the characteristic impedance of the continuously graded structure. Again, since $\mathbf{U}$ contains V_0 as a linear factor, the impedance is independent of the arbitrary applied voltage.

Since the impedance scales as $y_{\max}^{-1}$ in (27), it becomes convenient to express lengths in units of $y_{\max}$. In terms of such scaled lengths, $y_{\max}$ becomes unity, so that with the choice of $V_0 = 1.0$, a normalized impedance can be calculated as

$$f_g = Z_C / Z_0 = \left(\mathbf{U}_n^{\mathbf{T}} \mathbf{K}_n \mathbf{U}_n \right)^{-1} \tag{28}$$

where the potentials are expressed as fractions of the applied voltage and the Dirichlet matrix, $\mathbf{K}_n$, is now dimensionless.

The MATLAB® software package [3] and its Partial Differential Equation Toolbox [4] can be used to set up and solve the problems indicated by (13) and (22), using the finite element method. The MATLAB output, which includes both the solution vector and the Dirichlet matrix, can then be used to obtain the capacitance from (18) or (26), and the impedance from (20) or (27).

For problems described in terms of scaled lengths, it follows from (28) that the impedance is invariant to changes in the scale length parameter, $y_{\max}$. It depends only on the relative lengths and curvatures expressed by $\mathbf{K}_n$. Conversely, from (27), any specific transmission line geometry gives rise to a family of bends differing in impedance, as dictated by $y_{\max}$ and the resultant permittivity content. As a result, a relatively small number of problems spanning the range of realizable curvatures may suffice to characterize any specific cross section shape. We anticipate presenting an example of such a shape characterization in a future note.

3. Solutions of Canonical Problems

We begin by considering a problem for which an analytical solution exists in both curved and purely rectangular geometries—the parallel plate transmission line in which the plate width greatly exceeds the plate separation and fringe fields at the plate edges are neglected.

In [5], Baum develops an analytic solution for the impedance of an H-plane bend filled with a purely dielectric material whose permittivity varies continuously with the radius of curvature, as in (1). Since that solution neglects fringe fields, the only electric field component is directed normally to the conductor surfaces; and, from [6, Appendix A], we obtain

$$Z_C = Z_0\left(\frac{\Delta z}{y_{\max}\log_e(y_2/y_1)}\right) \qquad (29)$$

where the plate separation is Δz, the plate width is $(y_2 - y_1)$, and $0 < y_1 < y_2 \le y_{\max}$. By way of example, we choose $\Delta z = 0.05$, $y_1 = 0.1$, $y_2 = 0.6$, and $y_{\max} = 1.0$, obtaining $Z_C = 10.51\Omega$.

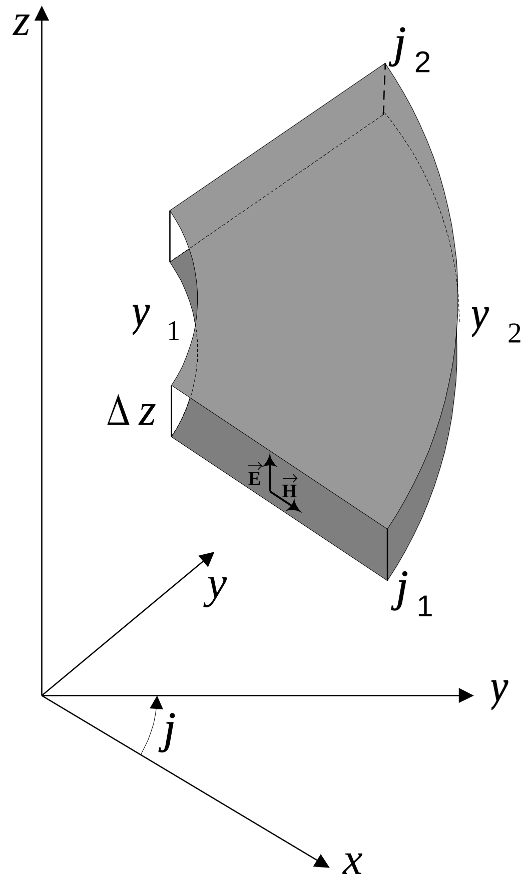

Figure 1. Parallel plate H-plane bend. There is rotational symmetry about the *z*-axis.

Neglecting fringe fields, the impedance of a straight parallel plate transmission line with the same cross section is given by

$$Z_C = Z_0\left(\frac{\Delta z}{\sqrt{e_r}(y_2 - y_1)}\right) \qquad (30)$$

where $e_r = e/e_0$ is the (constant) relative permittivity. By solving (30) for e_r, we find that the straight line will have the same characteristic impedance as the curved one, if $e_r = 12.83$. This choice thus

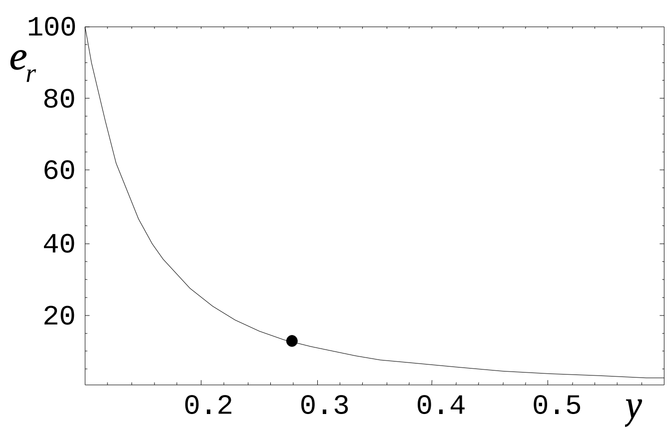

Figure 2. Relative permittivity for the parallel plate H-plane bend with $y_{\max} = 1.0$. The plotted point indicates the permittivity required for a matched straight section of transmission line.

becomes a design criterion for matching straight and curved sections of the transmission line. We also use it below in obtaining a numerical solution for the straight parallel plate problem.

To obtain finite element solutions for the curved and straight parallel plate problems, we define the two-dimensional problem domain to be the rectangular $\Delta y \times \Delta z$ area between the two conducting plates. We impose the arbitrary Dirichlet boundary conditions, $V = V_0 = 1.0$ on the top plate and $V = 0$ on the bottom plate. Along the y_1 and y_2 edges, we impose the Neumann condition, $\hat{n} \cdot \nabla V = 0$. This eliminates fringe fields by forcing the electric field there to be normal to the plates. For the curved plate geometry, we solve (22) for the potential and obtain the impedance from (27). For the straight plate geometry, we solve $\nabla \cdot (e_r \nabla V) = 0$, a two-dimensional rectangular version of (4) with the permittivity of free space factored out. An approach similar to that used to obtain (20) or (27) leads to

$$\boxed{Z_C = \frac{1}{vC} = Z_0 \left(\frac{V_0^2 \sqrt{e_r}}{\mathbf{U}^{\mathbf{T}} \mathbf{S}' \mathbf{U}} \right)} \tag{31}$$

where v is the velocity of propagation in the dielectric medium, C is the capacitance per unit length of transmission line, and V_0 is the potential difference between the conducting plates. Here, we have used $\mathbf{S}'$ in the vector-matrix expression to emphasize that this is not the same Dirichlet matrix as the one appearing in (17), (18), and (20). For both the curved and straight parallel plate transmission lines, the impedance obtained from the finite element solutions is 10.52 **Ω**, in agreement with the analytic results.

Next we consider coaxial geometries. For a straight coaxial line with uniform dielectric and conductors having circular cross sections, we have the familiar analytic expression

$$Z_C = Z_0 \frac{1}{2p\sqrt{e_r}} \log_e \left(\frac{b}{a} \right) \tag{32}$$

where a is the radius of the inner conductor and b is the radius of the outer conductor. Once again, (31) is used to obtain the impedance from the finite element solution. For a curved section of coax compensated by a dielectric as described by (1), there is no analytic solution. As was the case for the curved parallel plate transmission line, the finite element result is obtained from (27).

As an example, consider a curved section of coaxial cable with a = 0.025, b = 0.25, axis centered at (y , z) = (0.35, 0.25), and permittivity profile as described by (1), with $y_{max} = 1.0$.

The *z*-coordinate of the axis is completely arbitrary. As with the curved parallel plate, the choice of $y_{max} = 1.0$ effectively normalizes the radius of curvature, and in conjunction with the conductor radii, determines the range of permittivities within the coax (Figures 3 and 4).

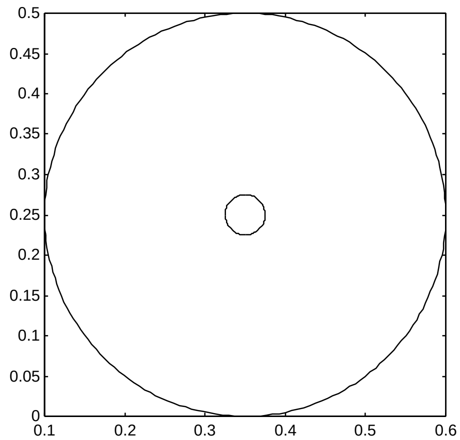

Figure 3. Geometry of a coaxial transmission line with circular cross section. For a curved section, the horizontal axis represents the radius of curvature.

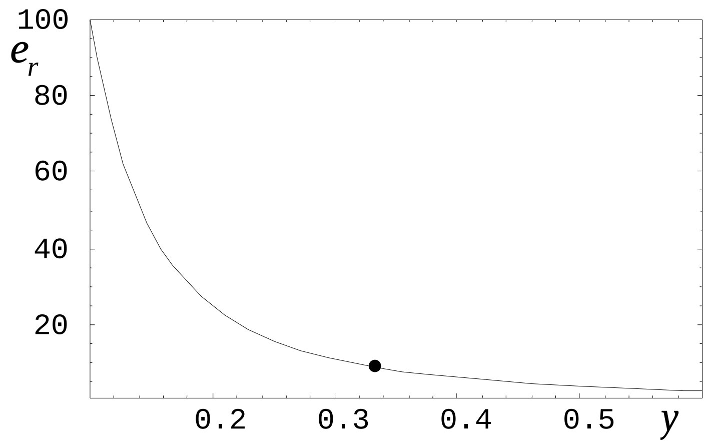

Figure 4. Relative permittivity for the coaxial H-plane bend with $y_{max} = 1.0$. The coaxial axis is on a plane of constant *z*, with radius of curvature, $y = 0.35$. The permittivity required for a straight coax of the same cross section and impedance is indicated by the plotted point at $y = 0.33$.

The impedance obtained from the finite element solution is 46.02 Ω. From (32), we can obtain an analytic result for the constant relative permittivity required to produce the same impedance in a straight section of coax having the same conductor size ratio (*b*/*a* = 10). The result is $e_r = 8.998$, a value that occurs in the curved coax at $y = 0.33$. If we use this relative permittivity in a finite element solution for the straight coax, (31) produces the consistent result, $Z_C = 45.96\Omega$. Thus, this choice of permittivity can be used to impedance-match straight and curved sections of the coaxial line.

We now consider coaxial geometries in which the center conductor has a circular cross section (diameter, *d*), while the outer conductor has a square cross section (length of side, *D*). For a straight section of such a transmission line filled with a uniform dielectric, [7] offers the closed form expression:

$$Z_C = \left(138\log_{10} r + 6.48 - 2.34A - 0.48B - .012C\right)\Big/\sqrt{e_r}$$

where

$$\begin{aligned} r &= D/d \\ A &= \left(1 + 0.405r^{-4}\right)\Big/\left(1 - 0.405r^{-4}\right) \\ B &= \left(1 + 0.163r^{-8}\right)\Big/\left(1 - 0.163r^{-8}\right) \\ C &= \left(1 + 0.067r^{-12}\right)\Big/\left(1 - 0.067r^{-12}\right) \end{aligned} \qquad (33)$$

Again, the finite element result for this straight coaxial geometry is obtained from (31). Similarly, if we replace this straight coax by a curved section compensated by a dielectric having the permittivity profile described by (1), we obtain a finite element result from (27).

We employ, as an example, a coax with the same parameters as used above, with $e_r = 8.998$, $D = 2b$ and $d = 2a$, so that $r = 10$. The geometry is that of Figure 3, with the outer conductor boundary replaced by its circumscribed square. For the straight coax, (33) produces $Z_C = 47.19\,\Omega$. The finite element result from (31) is 47.47 Ω. For the compensated curved coax, (27) gives the result, $Z_C = 47.23\,\Omega$. Once again, the finite element method produces a result consistent with a known closed-form solution. Moreover, for both straight and curved coaxial structures, the impedance of the circular geometry is comparable to that of the square geometry.

As a final example, we consider the layered approximation to a compensated bend in the square coaxial structure depicted in Figure 5. We retain the same parameters as in the previous square coax example, but partition the y -dimension into five equal-width dielectric bands. Each band is of width 0.1 and is assigned a constant relative permittivity equal to y^{-2} at its midpoint. In this way, the permittivity profile is a discrete approximation of (1). The finite element method is used to solve (13) by assigning the appropriate value of e_r to each of the five dielectric strip sub-domains. The approximate impedance, obtained from (20), is 47.70 Ω, in excellent agreement with the result for the continuously graded square coax.

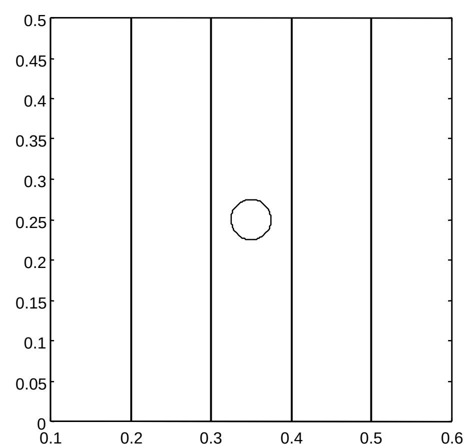

Figure 5. Geometry for a five-layer approximation to a graded coaxial bend.

4. Concluding Remarks

The derivation presented in this note provides a basis for use of finite element solutions of the potential equation to obtain the capacitance of curved transmission line sections in axially symmetric geometries. For systems in which the permittivity varies as the inverse square of the radius of curvature, TEM modes propagating in the azimuthal direction may be supported. In those cases, the approach described in this note can also be used to obtain a characteristic impedance for the transmission line structure. Numerical results indicate that even a rather coarse layered approximation of the ideally graded dielectric may closely reproduce this impedance.[†]

It should be noted that the approach developed here can be applied to any TEM transmission line geometry which can be described in two dimensions. The shape of the conductor cross sections is arbitrary. For curved sections, we must add the requirements of axial symmetry and a permittivity profile as in (1). The impedance is invariant to changes in scale length when that length is defined as the radius of curvature at which the relative permittivity is unity, y_{max}. On the other hand, if the cross section and radii of curvature are held constant, the impedance varies inversely with y_{max}.

In the examples presented here, we have applied the finite element method in two dimensions to both curved and straight transmission line sections. Since most practical systems will involve both curved and straight sections and junctions between the two, the interfaces between curved sections with non-uniform permittivity and straight sections with uniform permittivity must ultimately be considered.

[†] In Appendix A, we re-examine the more complex, multi-region stripline geometry originally treated in [6]. We find that the approximation used in [6] is in agreement with the current method.

Appendix A. Compensation of a Stripline Bend by a Layered Approximation of a Graded Dielectric Material

In [6], we presented experimental and numerical results for a stripline bend compensated by a layered approximation of graded dielectric material. There, we justified a straight strip approximation to the impedance of the bend by comparisons with related problem configurations and by comparison to a direct experimental measurement of the impedance. Here we re-examine that work in light of the derivation presented above.

Our finite element solution for the stripline in [6] employed an equivalent of (31) to calculate the impedance. There, we replaced the e_r of (31) with an effective permittivity obtained by averaging the permittivities of the dielectric materials surrounding the strip. Since the electric field was predominantly parallel to the dielectric interfaces, we argued that this averging approach was justifiable, as the strip and graded dielectric layers were like a parallel array of capacitors. The finite element model (Figure 6) defined a sub-domain for each dielectric material and solved the Laplace equation, $\nabla \cdot (e_r \nabla V) = 0$, in rectangular geometry, over all sub-domains. With an effective relative permittivity of 3.1, the calculated impedance was 27.93 Ω.

As a result of the expressions derived in this note, we no longer require the straight strip approximation. The correct potential equation for the axially symmetric geometry is $\nabla \cdot (y \ e_r \nabla u) = 0$, derived as (13) above. Assuming that the angular propagation velocity (in the j-direction) is approximately constant, the characteristic impedance is properly given by (20) as 28.57 Ω. Thus, for this problem, the straight strip approximation was a good approximation. Note that the impedance of the stripline bend would be 26.44 Ω, if filled with a continuously graded dielectric obeying (1). This follows from the solution of (22) and from (27).

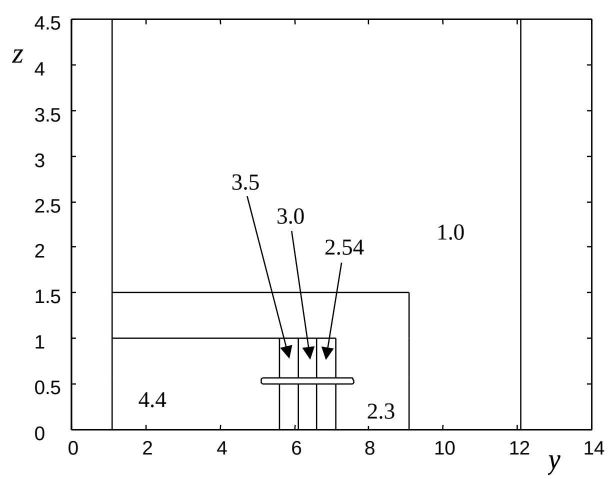

Figure 6. Finite element model of a five-layer graded dielectric bend. Relative permittivities used for the sub-domains are indicated.

References

1. P. P. Silvester and R. L. Ferrari, *Finite Elements for Electrical Engineers*, 2nd Ed., Cambridge University Press, 1990, pages 66-70.

2. C. E. Baum, *Azimuthal TEM Waveguides in Dielectric Media*, Sensor and Simulation Notes, Note 397, 31 March 1996.

3. *Using MATLAB*, Version 5, The MathWorks, Inc., Natick, MA 01760-1500, January 1997.

4. *Partial Differential Equation Toolbox User's Guide*, The MathWorks, Inc., Natick, MA 01760-1500, August 1995.

5. C. E. Baum, *Dielectric Body-of-Revolution Lenses with Azimuthal Propagation*, Sensor and Simulation Notes, Note 393, 9 March 1996.

6. W. S. Bigelow and E. G. Farr, *Minimizing Dispersion in a TEM Waveguide Bend by a Layered Approximation of a Graded Dielectric Material*, Sensor and Simulation Notes, Note 416, 5 January 1998.

7. *Reference Data for Engineers: Radio, Electronics, Computer, and Communications*, 8th Ed., SAMS, Prentice Hall Computer Publishing, Carmel, Indiana 46032, 1993, pages 29-22.

Acknowledgments

We would like to thank Mr. William B. Prather of the Air Force Research Laboratory for supporting the funding of this effort. We are also indebted to Dr. Carl E. Baum for the direction provided by his theoretical work in this area.